

New Results for the Correlation Functions of the Ising Model and the Transverse Ising Chain

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Received: 11 April 2009 / Accepted: 12 May 2009 / Published online: 23 May 2009
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Abstract In this paper we show how an infinite system of coupled Toda-type nonlinear differential equations derived by one of us can be used efficiently to calculate the time-dependent pair-correlations in the Ising chain in a transverse field. The results are seen to match extremely well long large-time asymptotic expansions newly derived here. For our initial conditions we use new long asymptotic expansions for the equal-time pair correlation functions of the transverse Ising chain, extending an old result of T.T. Wu for the 2d Ising model. Using this one can also study the equal-time wavevector-dependent correlation function of the quantum chain, a.k.a. the q -dependent diagonal susceptibility in the 2d Ising model, in great detail with very little computational effort.

Keywords Ising model · Transverse Ising chain · Correlation function · Time-dependence · Susceptibility

1 Introduction

In recent years there has been much interest in the properties of low-dimensional quantum systems. It seems to be worthwhile, therefore, to present a new algorithm to efficiently and accurately calculate a very large number of time-dependent pair correlations in the bulk for the premier example of such systems, namely for the transverse Ising chain.¹ Many results

¹The transverse Ising chain [1] is a special case of the XY model first introduced by Nambu [2] in 1950 for the isotropic zero-field case and generalized by Lieb, Schultz and Mattis [3] and Katsura [4].

Supported in part by the National Science Foundation under grant PHY 07-58139 and by the Australian Research Council under Project ID: LX0989627.

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have been derived before for this model,² but the foremost method is still to calculate very large determinants for each data point, using either the appropriate determinant for the open chain with both spins far from the boundary [10], or the infinite McCoy–Barouch–Abraham determinant [11] for the closed chain.

In this paper we shall use a set of coupled nonlinear differential-difference equations derived by one of us [12, 13], related to similar identities for the planar Ising model [14, 15]. From these we can obtain multiple millions of data points in one single run. In spite of what some colleagues have told us—this work is partly an answer to their unbelief—our nonlinear equations are highly effective for numerical computations as we shall demonstrate. We shall concentrate on the zero-temperature case,³ for which the initial conditions coincide with the diagonal correlations in the two-dimensional Ising model.

Therefore, we shall first discuss in Sect. 2 how to accurately calculate these diagonal correlations. For smaller separations we can use the algorithms of Jimbo and Miwa [42, 43] or of Witte [44]. For larger separations we need several more terms in the old asymptotic expansion of T.T. Wu [45, 46], which we shall derive as a new result.

In Sect. 3 we introduce the transverse-field Ising chain. We shall also present new results for the asymptotic expansions for its correlations as a function of time. In Sect. 4 we shall give details of how we solved the correlations numerically and show how well the numerical results agree with the asymptotic expansions derived in Sect. 3. We conclude with a few remarks in Sect. 5.

2 Results for the Two-dimensional Ising Model

In this section we shall present results for the diagonal pair correlations of the two-dimensional Ising model on the square lattice and its dual model. These models are described by the reduced interaction energies (with factor $\beta = 1/k_B T$ absorbed)

$$-\beta\mathcal{E} = \sum_m \sum_n (K_1 \sigma_{m,n} \sigma_{m+1,n} + K_2 \sigma_{m,n} \sigma_{m,n+1}), \quad (1)$$

and

$$-(\beta\mathcal{E})^* = \sum_m \sum_n (K_1^* \sigma_{m,n} \sigma_{m+1,n} + K_2^* \sigma_{m,n} \sigma_{m,n+1}). \quad (2)$$

It is convenient to introduce the following short-hand notations for the interaction parameters:

$$t_1 \equiv \tanh K_1 = e^{-2K_2^*}, \quad t_2 \equiv \tanh K_2 = e^{-2K_1^*}, \quad (3)$$

and

$$\begin{aligned} S_1 &\equiv \sinh(2K_1) = 1/\sinh(2K_2^*) = 1/S_2^*, \\ S_2 &\equiv \sinh(2K_2) = 1/\sinh(2K_1^*) = 1/S_1^*. \end{aligned} \quad (4)$$

²We shall not consider the time-dependent zz correlations [5–9], which being only 2-by-2 determinants do not involve “Jordan–Wigner strings” of fermion operators.

³The case of finite temperature should work out fine also according to a calculation we have done with approximate initial conditions, utilizing the exponential decay with separation in the initial conditions. The case of infinite temperature has been studied before in great detail [16–23]. We shall use some of the older results for zero temperature [11, 24–31]. Finally, there are several partial results for finite temperature worth mentioning, e.g. [32–41].

We shall restrict the elliptic modulus

$$k \equiv S_1 S_2 \tag{5}$$

to $0 \leq k \leq 1$, as the case $k > 1$ is included via the dual model,⁴ defining

$$C(m, n) \equiv \langle \sigma_{0,0} \sigma_{m,n} \rangle = \sum_{\{\sigma\}} \sigma_{0,0} \sigma_{m,n} e^{-\beta \mathcal{E}} / \sum_{\{\sigma\}} e^{-\beta \mathcal{E}}, \tag{6}$$

$$C^*(m, n) \equiv \langle \sigma_{0,0} \sigma_{m,n} \rangle^* = \sum_{\{\sigma\}} \sigma_{0,0} \sigma_{m,n} e^{-\beta \mathcal{E}^*} / \sum_{\{\sigma\}} e^{-\beta \mathcal{E}^*}. \tag{7}$$

As already said before, for the initial conditions of the quantum chain, we need to calculate $C(m, n)$ and $C^*(m, n)$ for $m = n$. This can be done using the quadratic recurrence relations of Jimbo and Miwa [42, 43] or Witte [44].

When $T \neq T_c$, we can use the well-known Toeplitz determinants [46, 47]. It is, however, more efficient to use the equations provided by Jimbo and Miwa [42, 43]. Denoting $C^*(n, n) \equiv A_n$ and $C(n, n) \equiv C_n$, these can be rewritten as:

$$B_{n+1} = -(kA_n^+ B_n^+ + k^{-1} A_n^- B_n^-) / ((2n + 3)A_n), \tag{8}$$

$$C_{n+1}^\pm = (A_{n+1} C_n^\pm - C_n A_n^\pm) / (k^{\pm 1} A_n), \tag{9}$$

$$D_{n+1}^\pm = (A_{n+1} D_n^\pm + C_n B_n^\pm) / A_n, \tag{10}$$

$$C_{n+1} = -(C_{n+1}^+ D_{n+1}^+ + C_{n+1}^- D_{n+1}^-) / ((2n + 1)A_n), \tag{11}$$

$$A_{n+1}^\pm = (A_{n+1} A_n^\pm - B_{n+1} C_{n+1}^\pm) / A_n, \tag{12}$$

$$B_{n+1}^\pm = (k^{\pm 1} A_{n+1} B_n^\pm + B_{n+1} D_{n+1}^\pm) / A_n, \tag{13}$$

$$A_{n+2} = (A_{n+1}^2 - B_{n+1} C_{n+1}) / A_n, \tag{14}$$

which can be solved iteratively, in the above order for $n = 0, 1, \dots$, starting from the initial conditions

$$\begin{aligned} A_0 = B_0 = C_0 = 1, \quad A_1 = 2E(k)/\pi, \\ B_0^+ = D_0^- = k', \quad C_0^+ = A_0^- = 1/k', \quad D_0^+ = C_0^- = 0, \\ A_0^+ = 2(2E(k) - K(k))/(\pi k'), \quad B_0^- = 2k'(K(k) - E(k))/\pi, \end{aligned} \tag{15}$$

where $K(k)$ and $E(k)$ are the usual complete elliptic integrals, $k' = \sqrt{1 - k^2}$. From these equations one can calculate systematically more A_n 's and C_n 's by iteration, keeping sufficiently many digits, until the asymptotic regime is reached.

The required asymptotic expansions can be obtained from the Painlevé VI equation (PVI) of Jimbo and Miwa [42, 43], extending the expansions of Wu [45]—or also (2.46) and (3.27) of Chap. XI of [46]—beyond second order. For solving PVI iteratively, we need information from the leading terms in the low- and high-temperature expansions of the connected pair correlations.

⁴The case $k < 0$ follows by simple gauge symmetry, but will not be used.

The high-temperature expansion of $C(m, n)$ is well-known and is often treated in graduate courses. Without loss of generality we can take $m, n \geq 0$ and give the leading high-temperature term as

$$C(m, n) \approx \frac{(m+n)!}{m!n!} t_1^m t_2^n \approx \frac{(m+n)!}{m!n!} \frac{S_1^m S_2^n}{2^{m+n}}, \tag{16}$$

where the combinatorial factor counts the number of staircase walks from $(0, 0)$ to (m, n) on the square lattice. For the leading term in the low-temperature expansion of the connected pair correlation

$$C_c^*(m, n) = C^*(m, n) - (1 - k^2)^{1/4}, \quad k \equiv S_1 S_2 = \frac{1}{S_1^* S_2^*}, \tag{17}$$

subtracting the square of the spontaneous magnetization, we can use (10a), (10b) or (10c) of [15]. When doing so we need to replace $M \rightarrow n, N \rightarrow m$, as currently it is more common to use the horizontal coordinate as the first one. Rewriting (10b) of [15] as

$$\begin{aligned} & \left(C^*(m-1, n) C^*(m+1, n) - C^*(m, n)^2 \right) \\ & + S_2^2 \left(C(m, n-1) C(m, n+1) - C(m, n)^2 \right) = 0, \end{aligned} \tag{18}$$

we can ignore the contributions of $C_c^*(m, n)$ and $C_c^*(m+1, n)$ and use $k \rightarrow 0$. We arrive at

$$C_c^*(m-1, n) \approx S_2^2 \left(C(m, n)^2 - C(m, n-1) C(m, n+1) \right), \tag{19}$$

or equivalently

$$C_c^*(m, n) \approx S_2^2 \left(C(m+1, n)^2 - C(m+1, n-1) C(m+1, n+1) \right), \tag{20}$$

with the leading order solution

$$C_c^*(m, n) \approx \frac{(m+n)! (m+n+1)!}{m! (m+1)! n! (n+1)!} \frac{S_1^{2m+2} S_2^{2n+2}}{2^{2m+2n+2}}. \tag{21}$$

Here the combinatorial coefficient can be recognized as a Narayana number, as it counts all staircase polygons needed in the leading order of the low-temperature expansion.⁵ Setting $m = n$, we get⁶

$$C(n, n) \approx \frac{(2n)!}{(n!)^2} \frac{k^n}{2^{2n}}, \quad C_c^*(n, n) \approx \frac{(2n)! (2n+1)!}{(n!)^2 [(n+1)!]^2} \frac{k^{2n+2}}{2^{4n+2}}. \tag{22}$$

⁵We thank Dr. Xavier Viennot for pointing this out at the Dunk Island Conference of 2005. The q -analogues of the Narayana numbers were already studied by MacMahon four decades before Narayana rediscovered them, see [48] and references cited therein.

⁶These results have been advocated by Dr. Ranjan Kumar Ghosh [49] as initial conditions for the derivation of the first few terms in the high- and low-temperature expansions using PVI, with the second result in (22) presented as a conjecture. However, there is a much more efficient way to derive such expansions [50, 51].

Following Jimbo and Miwa, we define

$$\begin{aligned}
 t &\equiv \frac{1}{k^2}, \quad \sigma_n \equiv t(t-1) \frac{d}{dt} \ln C(n, n) - \frac{1}{4}t, \\
 \sigma_n^* &\equiv t(t-1) \frac{d}{dt} \ln C^*(n, n) - \frac{1}{4},
 \end{aligned}
 \tag{23}$$

and then σ_n and σ_n^* both satisfy the PVI equation

$$\begin{aligned}
 &\left[t(t-1) \frac{d^2 \sigma_n}{dt^2} \right]^2 - n^2 \left[(t-1) \frac{d\sigma_n}{dt} - \sigma_n \right]^2 \\
 &+ 4 \frac{d\sigma_n}{dt} \left[(t-1) \frac{d\sigma_n}{dt} - \sigma_n - \frac{1}{4} \right] \left[t \frac{d\sigma_n}{dt} - \sigma_n \right] = 0.
 \end{aligned}
 \tag{24}$$

Substituting

$$C(n, n) = \frac{t^{-n/2}}{\sqrt{\pi n}(1-1/t)^{1/4}} \exp \left(\sum_{j=1}^m \sum_{s=0}^{\lfloor j/2 \rfloor} \frac{p_{j,s} x^{j-2s}}{n^j} \right)
 \tag{25}$$

for increasing values of $m = 1, 2, \dots$, while defining

$$x = \frac{t+1}{t-1} = \frac{1+k^2}{1-k^2}
 \tag{26}$$

($x = x'_3$ on page 256 of [46]), we can easily solve all the $p_{j,s}$'s. Whenever m is even, we also need (22) in the limit $k \rightarrow 0$ or $x \rightarrow 1$ in order to extract $p_{m,m/2}$. Similarly, substituting

$$C_c^*(n, n) = \frac{t^{-n-1}}{2\pi n^2(1-1/t)^2} \exp \left(\sum_{j=1}^m \sum_{s=0}^{\lfloor j/2 \rfloor} \frac{p_{j,s}^* x^{j-2s}}{n^j} \right)
 \tag{27}$$

for increasing values of $m = 1, 2, \dots$, with the same x ($x = -x'_3$ in [46] now), we obtain the $p_{j,s}^*$'s using (22) in addition for the $p_{m,m/2}^*$ whenever m is even. Using short Maple programs we went up to $m = 20$ and ended with

$$p_{20,10} = \frac{74074237647505}{8388608}, \quad p_{20,10}^* = \frac{673835095036826977}{10485760}.
 \tag{28}$$

We have used all these results, but here we only give the results up to $m = 10$, i.e.

$$\begin{aligned}
 C(n, n) &= \frac{k^n}{\sqrt{\pi n}(1-k^2)^{1/4}} \exp \left(-\frac{x}{8n} + \frac{(x^2-1)}{16n^2} - \frac{x(25x^2-27)}{384n^3} \right. \\
 &+ \frac{(x^2-1)(13x^2-5)}{128n^4} - \frac{x(1073x^4-1830x^2+765)}{5120n^5} \\
 &+ \frac{(x^2-1)(412x^4-425x^2+61)}{768n^6} \\
 &- \frac{x(375733x^6-886725x^4+660723x^2-150003)}{229376n^7} \\
 &+ \left. \frac{(x^2-1)(23797x^6-40211x^4+18055x^2-1385)}{4096n^8} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{x(55384775x^8 - 167281524x^6 + 179965314x^4 - 79479684x^2 + 11415087)}{2359296n^9} \\
 & + \frac{(x^2 - 1)(2180461x^8 - 5127404x^6 + 3945946x^4 - 1048244x^2 + 50521)}{20480n^{10}} - \dots \quad (29)
 \end{aligned}$$

and

$$\begin{aligned}
 C_c^*(n, n) = & \frac{k^{2n+2}}{2\pi n^2(1-k^2)^2} \exp\left(-\frac{7x}{4n} + \frac{17x^2 - 10}{8n^2} - \frac{901x^3 - 783x}{192n^3}\right. \\
 & + \frac{899x^4 - 1062x^2 + 194}{64n^4} - \frac{131411x^5 - 196770x^3 + 66375x}{2560n^5} \\
 & + \frac{83591x^6 - 151767x^4 + 75033x^2 - 6730}{384n^6} \\
 & - \frac{17052139x^7 - 36416187x^5 + 23770797x^3 - 4402125x}{16384n^7} \\
 & + \frac{11282939x^8 - 27723492x^6 + 22515930x^4 - 6419700x^2 + 344834}{2048n^8} \\
 & - \frac{37620804281x^9 - 104587369452x^7 + 101707083486x^5 - 39418182684x^3 + 4677930225x}{1179648n^9} \\
 & \left. + \frac{2049064082x^{10} - 6360721245x^8 + 7210080180x^6 - 3544939170x^4 + 670637250x^2 - 24119050}{10240n^{10}} - \dots\right). \quad (30)
 \end{aligned}$$

At this point we have an effective way of calculating the connected diagonal correlation functions of the square-lattice Ising model and their logarithms, by iteration up to a certain distance using many-digit precision as this procedure is unstable, and by asymptotic expansion for larger distances. This way we also have the zero-time zero-temperature xx -correlations of the transverse Ising chain for $0 < k < 1$. As we will need very large numbers of $\log C(n, n)$ and $\log C_c^*(n, n)$ at fixed k we will have to do the sums over s only once saving computation time.

For the self-dual case $k = 1$, with $C^*(n, n) = C_c^*(n, n) = C(n, n)$ there is a simple formula, which was already known to Onsager [52], discussed in detail by McCoy and Wu [46], and implemented by us as:

$$\log C(0, 0) = 0, \quad \log C(n, n) = \log C(n - 1, n - 1) + r_n, \quad (31)$$

$$r_1 = \log(2/\pi), \quad r_{n+1} = r_n - \log\left(1 - 1/(4n^2)\right). \quad (32)$$

For this case we also have the large- n asymptotic expansion [45–47], but we did not need it for this paper.

When k is close to 1, the results are described by Painlevé III or V (PIII or PV) scaling limit results [42, 43, 47, 53]. Vaidya and Tracy [25] have analyzed the time-dependent pair correlations of the transverse Ising chain in this scaling limit. McCoy, Perk and

Shrock [26, 27] found the Painlevé II scaling limit describing the crossover from the space-like to the time-like regime for this quantum chain model.

3 Quantum Ising Chain

From now on we shall consider equilibrium bulk properties of the ferromagnetic Ising chain in a positive transverse field. The sites are labeled by integers j , (with $-\mathcal{N} \leq j \leq \mathcal{N}$, and $\mathcal{N} \rightarrow \infty$ in the thermodynamic limit). We shall take the usual spin- $\frac{1}{2}$ operator basis for the quantum chain

$$\begin{aligned}
 \sigma_j^x &\equiv \cdots \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \overbrace{\left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right)}^{j\text{-th}} \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \cdots, \\
 \sigma_j^y &\equiv \cdots \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \cdots, \\
 \sigma_j^z &\equiv \cdots \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \otimes \cdots.
 \end{aligned}
 \tag{33}$$

The spin operators obey the usual Schrödinger time-dependence in the Heisenberg picture

$$\sigma_j^\alpha(t) \equiv e^{i\mathcal{H}t} \sigma_j^\alpha e^{-i\mathcal{H}t} \quad (\text{in units for which } \hbar \equiv 1, \alpha = x, y, z),
 \tag{34}$$

with Hamiltonian \mathcal{H} (or dual Hamiltonian \mathcal{H}^*)

$$\mathcal{H} = -\frac{1}{2} \sum_{j=-\infty}^{\infty} (J\sigma_j^x \sigma_{j+1}^x + B\sigma_j^z), \quad \mathcal{H}^* = -\frac{1}{2} \sum_{j=-\infty}^{\infty} (B\sigma_j^x \sigma_{j+1}^x + J\sigma_j^z),
 \tag{35}$$

with $J, B > 0$. The dual chain corresponds to the interchange of J and B .⁷ The pair correlation function

$$X_n(t) \equiv \langle \sigma_j^x(t) \sigma_{j+n}^x \rangle \equiv \frac{\text{Tr} (e^{i\mathcal{H}t} \sigma_j^x e^{-i\mathcal{H}t} \sigma_{j+n}^x e^{-\beta\mathcal{H}})}{\text{Tr} (e^{-\beta\mathcal{H}})}
 \tag{36}$$

and the dual $X_n^*(t)$, with \mathcal{H} replaced by \mathcal{H}^* , satisfy [12, 13]

$$\begin{cases} X_n(t) \ddot{X}_n(t) - \dot{X}_n(t)^2 = B^2 (X_{n-1}^*(t) X_{n+1}^*(t) - X_n^*(t)^2), \\ X_n^*(t) \ddot{X}_n^*(t) - \dot{X}_n^*(t)^2 = J^2 (X_{n-1}(t) X_{n+1}(t) - X_n(t)^2), \end{cases}
 \tag{37}$$

⁷Strictly spoken, the dual positions are at the half-integers, but we can relabel them to the integers. Also, as the bulk thermodynamic limit for pair correlations has been well-established and is independent of boundary conditions, we have formally replaced \mathcal{N} by ∞ in the bounds of the summations in (35). This thermodynamic limit is to be understood to be part of the pair correlation’s definition in the following.

for all inverse temperatures $\beta = 1/k_B T$. At the critical field $B = J$ this reduces to [12, 13]

$$X_n(t)\ddot{X}_n(t) - \dot{X}_n(t)^2 = J^2(X_{n-1}(t)X_{n+1}(t) - X_n(t)^2). \tag{38}$$

For real times t these equations represent a discrete nonlinear generalization of a system of hyperbolic partial differential equations. Therefore, we expect the initial-value problem to be stable and this is indeed the case for sufficiently small time steps. On the other hand, for imaginary times t , as needed for the calculation of the susceptibility, we have an elliptic system and we should treat these equations as a boundary value problem or work with an increasingly high number of digits as $|t|$ increases, as was done before for the (Euclidian) two-dimensional Ising model, first for the uniform case [54–57] and later for quasi-periodic cases [58–61].⁸ In fact, the susceptibility for the transverse Ising model has been calculated in the mid 1980s as an extremely anisotropic limit of the corresponding 2d Ising result [54–57].

Before describing the algorithm used, we shall first derive some long-time asymptotic expansions, extending the results on the last page of [11], while also implicitly correcting several misprints there. We start with adopting the results (2.1) and (2.7) in [25] to our case.⁹ For $T = 0$,

$$X_R(t) = \left(1 - \frac{B}{J}\right)^{1/4} \exp\left(-\sum_{n=1}^{\infty} F^{(2n)}\right), \quad (0 < B < J), \tag{39}$$

$$= \left(1 - \frac{J}{B}\right)^{1/4} X^{(2n-1)} \exp\left(-\sum_{n=1}^{\infty} F^{(2n)}\right), \quad (0 < J < B), \tag{40}$$

where

$$\begin{aligned} F_{>}^{(2n)}(R, t) &= F_{<}^{(2n)}(R, t) = F^{(2n)}(R, t) \\ &= \frac{1}{2n} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{2n} \prod_{j=1}^{2n} \left(L(\phi_j)M(\phi_j, \phi_{j+1})\right), \end{aligned} \tag{41}$$

and

$$\begin{aligned} X^{(2n-1)}(R, t) &= \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_{2n-1} \\ &\times \left[\prod_{j=1}^{2n-2} \left(L(\phi_j)M(\phi_j, \phi_{j+1})\right) \right] L(\phi_{2n-1})\bar{M}(\phi_{2n-1}, \phi_1). \end{aligned} \tag{42}$$

Here,

$$L(\phi) \equiv \frac{\exp\left(-iR\phi - it\lambda(\phi)\right)}{4\pi\lambda(\phi)}, \tag{43}$$

⁸These latter cases could be done because the general quadratic identities of [12] apply also to Z-invariant cases [62] with further implications for correlations and susceptibilities [47, 63–65].

⁹There are misprints in (1.2) and (1.3) of [25], which differ from the actual definitions used in [11, 18, 20, 22, 25]. The difference means that the results change to their complex conjugates, or equivalently to the replacement $t \rightarrow -t$.

$$M(\phi, \phi') \equiv \frac{\lambda(\phi) - \lambda(\phi')}{\sin(\frac{1}{2}(\phi + \phi'))} = \frac{4JB \sin(\frac{1}{2}(\phi - \phi'))}{\lambda(\phi) + \lambda(\phi')}, \tag{44}$$

$$\bar{M}(\phi, \phi') \equiv 2B \cos\left(\frac{1}{2}(\phi - \phi')\right), \tag{45}$$

$$\lambda(\phi) \equiv \sqrt{J^2 + B^2 - 2JB \cos \phi}. \tag{46}$$

It is easily checked that the integrands in (41) and (42) are periodic modulo 2π in all the ϕ_j . Therefore, before applying the stationary phase method for large t , we can shift the integration bounds and consider the stationary phase points $\phi_j = 0$ and $\phi_j = \pi \equiv -\pi$ as internal point for the integration over ϕ_j . Expanding the integrands of (41) and (42) in power series both in ϕ_j and in $\phi_j - \pi$ for all j , keeping only the quadratic terms in the exponentials of the $L(\phi_j)$'s, we can then use

$$\int_{-\infty}^{+\infty} d\phi e^{\pm i\alpha\phi^2} \phi^{2n-1} = 0, \quad \int_{-\infty}^{+\infty} d\phi e^{\pm i\alpha\phi^2} \phi^{2n} = e^{\pm(2n+1)i\pi/4} \frac{\Gamma(n + \frac{1}{2})}{\alpha^{n+\frac{1}{2}}} \tag{47}$$

for $\alpha > 0$. The leading terms come from $X^{(1)}$ and $F^{(2)}$ and to leading order one finds

$$X^{(2n+1)} \approx (F^{(2)})^n X^{(1)}, \quad F^{(2n)} \approx \frac{1}{n} (F^{(2)})^n, \tag{48}$$

as the stationary phase points contributing are alternately 0 and π , starting with $\phi_1 = 0$ or π . In higher orders of $1/t$ more and more non-alternating combinations come in and it soon becomes very tedious and unrepresentable. We shall, therefore, only give the results for $R = 0$ in (39) and (40) here.

We use the abbreviations

$$k = \frac{B}{J}, \quad \bar{t} = Jt \quad (\text{for } B < J), \quad k = \frac{J}{B}, \quad \bar{t} = Bt \quad (\text{for } B > J). \tag{49}$$

Then, for $B < J$ and $t \rightarrow \infty$,

$$\begin{aligned} X_0(t) = & (1 - k^2)^{1/4} \left(1 + \frac{k e^{-2i\bar{t}}}{2\pi \sqrt{1 - k^2} \bar{t}} \right. \\ & + \frac{ik(4 - 3k^2)e^{-2i\bar{t}}}{8\pi(1 - k^2)^{3/2}\bar{t}^2} - \frac{e^{-2i(1-k)\bar{t}}}{8\pi(1 - k)^2\bar{t}^2} - \frac{e^{-2i(1+k)\bar{t}}}{8\pi(1 + k)^2\bar{t}^2} \\ & + \frac{(4 - 60k^2 + 80k^4 - 33k^6)e^{-2i\bar{t}}}{64\pi k(1 - k^2)^{5/2}\bar{t}^3} \\ & + \frac{i(1 - 9k + k^2)e^{-2i(1-k)\bar{t}}}{32\pi k(1 - k)^3\bar{t}^3} - \frac{i(1 + 9k + k^2)e^{-2i(1+k)\bar{t}}}{32\pi k(1 + k)^3\bar{t}^3} \\ & \left. - \frac{ik^2(2 + k^2)e^{-4i\bar{t}}}{32\pi^2(1 - k^2)^2\bar{t}^3} + \dots \right), \tag{50} \end{aligned}$$

whereas, for $B > J$ and $t \rightarrow \infty$,

$$X_0(t) = (1 - k^2)^{1/4} \left(\frac{e^{-i(1-k)\bar{t} - \pi i/4}}{\sqrt{2\pi}(k(1 - k)\bar{t})^{1/2}} + \frac{e^{-i(1+k)\bar{t} + \pi i/4}}{\sqrt{2\pi}(k(1 + k)\bar{t})^{1/2}} \right)$$

$$\begin{aligned}
 & - \frac{(1 - 3k + k^2)e^{-i(1-k)\bar{t} + \pi i/4}}{8\sqrt{2\pi}(k(1 - k)\bar{t})^{3/2}} - \frac{(1 + 3k + k^2)e^{-i(1+k)\bar{t} - \pi i/4}}{8\sqrt{2\pi}(k(1 + k)\bar{t})^{3/2}} \\
 & - \frac{3(3 - 10k + 17k^2 - 10k^3 + 3k^4)e^{-i(1-k)\bar{t} - \pi i/4}}{128\sqrt{2\pi}(k(1 - k)\bar{t})^{5/2}} \\
 & - \frac{3(3 + 10k + 17k^2 + 10k^3 + 3k^4)e^{-i(1+k)\bar{t} + \pi i/4}}{128\sqrt{2\pi}(k(1 + k)\bar{t})^{5/2}} \\
 & - \frac{k^{3/2} e^{-i(3-k)\bar{t} + \pi i/4}}{4(2\pi)^{3/2}(1 - k)^2(1 + k)^{1/2} \bar{t}^{5/2}} \\
 & - \frac{k^{3/2} e^{-i(3+k)\bar{t} - \pi i/4}}{4(2\pi)^{3/2}(1 + k)^2(1 - k)^{1/2} \bar{t}^{5/2}} + \dots \Big). \tag{51}
 \end{aligned}$$

The lowest order terms that decay as $t^{-1/2}$ and t^{-1} have been given before in (5.14) of [11], and they agree after one corrects some misprints in [11].

At the critical field $B = J$, we can use the Painlevé V equation of [26] to obtain many terms in the long-time asymptotic expansion of $X_0(t)$. Writing

$$X_0(t) = e^{x^2/8} \tau_0(x), \quad \sigma_0(x) = x \frac{d \log \tau_0(x)}{dx}, \quad x \equiv 2iJt, \tag{52}$$

we must solve

$$\left(x \frac{d^2 \sigma}{dx^2} \right)^2 + 4 \left(x \frac{d\sigma}{dx} - \sigma \right) \left[x \frac{d\sigma}{dx} + \left(\frac{d\sigma}{dx} \right)^2 - \sigma \right] = 0, \tag{53}$$

requiring the large- x asymptotic behaviors

$$\sigma_0(x) \approx -\frac{x^2}{4} + z\sqrt{x} + \sum_{n=0}^N \sum_{m=0}^{\lfloor \frac{1}{2}n+1 \rfloor} c_{n,m} \frac{z^{n+2-2m}}{x^{n/2}}, \tag{54}$$

and

$$\log \tau_0(x) \approx -\frac{x^2}{8} + \log \frac{A}{(x/2)^{1/4}} + \sum_{n=1}^N \sum_{m=0}^{\lfloor \frac{1}{2}n \rfloor} d_{n,m} \frac{z^{n-2m}}{x^{n/2}}, \tag{55}$$

where [26, 66]

$$x \rightarrow \infty, \quad z \equiv -\frac{1}{\sqrt{2\pi}} e^{-x}, \quad A = 2^{1/12} e^{3z'(-1)}. \tag{56}$$

(Here we use the more common real definition of A without the $e^{-i\pi/8}$ factor of [26]. This factor is here supplied by the $i^{1/4}$ of $x^{1/4}$.)

We can solve the coefficients $c_{n,m}$ recursively from the sets of linear equations obtained by using the leading term of order $O(x^{(5-N)/2})$ when substituting (54) (with $N = 0, 1, \dots$, in succession) into (53). After each step in this process there are two undetermined coefficients left that are determined in the next two steps. The coefficients $d_{n,m}$ follow recursively for

$n = 3, 4, \dots$, from

$$(n - 2m) d_{n,m} = -c_{n-2,m} - \frac{1}{2}(n - 2) d_{n-2,m-1}, \tag{57}$$

using also

$$d_{n,0} = -\frac{1}{n}, \quad d_{2,1} = 0, \quad d_{2n,n} = -\frac{c_{2n,n+1}}{n}. \tag{58}$$

The final result, after exponentiating $\log \tau_0(x)$, is

$$\begin{aligned} X_0(t) = & \frac{A}{(x/2)^{1/4}} \left(1 - \frac{z}{x^{1/2}} + \frac{9z}{8x^{3/2}} - \frac{2z^2 - 1}{8x^2} - \frac{297z}{2^7 x^{5/2}} \right. \\ & + \frac{15z^2}{16x^3} + \frac{7587z}{2^{10} x^{7/2}} - \frac{489z^2 - 81}{2^7 x^4} + \frac{1024z^3 - 1027035z}{2^{15} x^{9/2}} \\ & + \frac{9387z^2}{2^9 x^5} - \frac{76800z^3 - 43594695z}{2^{18} x^{11/2}} \\ & - \frac{851427z^2 - 90072}{2^{13} x^6} + \frac{9094144z^3 - 4418168445z}{2^{22} x^{13/2}} \\ & + \frac{22520925z^2}{2^{15} x^7} - \frac{529007616z^3 - 260700970635z}{2^{25} x^{15/2}} \\ & + \frac{768z^4 - 1368815805z^2 + 108135000}{2^{18} x^8} \\ & + \frac{258931316736z^3 - 139999291654995z}{2^{31} x^{17/2}} \\ & - \frac{59904z^4 - 47100085335z^2}{2^{20} x^9} \\ & - \frac{17114655467520z^3 - 10543684346529075z}{2^{34} x^{19/2}} \\ & + \frac{24846336z^4 - 14491877193315z^2 + 908002224000}{2^{25} x^{10}} \\ & + \frac{2469862452602880z^3 - 1758124895330287575z}{2^{38} x^{21/2}} \\ & - \frac{1113772032z^4 - 616117763829645z^2}{2^{27} x^{11}} \\ & - \frac{195069658230835200z^3 - 160841975585736493125z}{2^{41} x^{23/2}} \\ & \left. + O(x^{-12}) \right). \tag{59} \end{aligned}$$

The first line of the above equation agrees with (43) in [26]. Here we have given a more efficient method to extend the asymptotic expansion, using real rational coefficients only.¹⁰

¹⁰Replacing the term $z\sqrt{x}$ in (54) by $-4Kx + (z + K/z)\sqrt{x}$ and summing m from 0 to $n + 2$, we can easily extend also (39) in [26], correcting two misprints in the highest order terms there in the process. One must then identify $K = -iab$ and $z = be^{-is - i\pi/4}$.

4 Numerical Results for Quantum Ising Chain

At zero temperature, the initial values relate to the diagonal correlation function $\langle \sigma_{00} \sigma_{nn} \rangle$ of the two-dimensional Ising model discussed in detail in Sect. 2. More precisely,

$$X_n(0) = \langle \sigma_{00} \sigma_{nn} \rangle, \quad \dot{X}_n(0) = \dot{X}_0(0) \delta_{n0} = -iB \langle \sigma_0^z \rangle \delta_{n0}. \tag{60}$$

Here, $k = J/B = S_1 S_2 < 1$ corresponds to the 2d Ising model with $T > T_c$, while the case $k = B/J = S_1^* S_2^* < 1$ corresponds to $T < T_c$. The initial value of the first time-derivative $\dot{X}_n(0)$ vanishes for $n \neq 0$ and for $n = 0$ it is given by the z -magnetization of the transverse Ising chain [1, 4, 67],

$$\langle \sigma_0^z \rangle = \frac{2}{\pi} E(k), \quad k = \frac{J}{B} \leq 1, \tag{61}$$

$$\langle \sigma_0^z \rangle = \frac{2}{\pi k} \left(E(k) - (1 - k^2) K(k) \right), \quad k = \frac{B}{J} \leq 1, \tag{62}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind. For $J = B$, we have the simple results, see e.g. [26, 46],

$$X_n(0) = \left(\frac{2}{\pi} \right)^n \prod_{l=1}^{n-1} \left(1 - \frac{1}{4} l^2 \right)^{l-n}, \quad \dot{X}_n(0) = -iB \frac{2}{\pi} \delta_{n0}. \tag{63}$$

We have tacitly assumed $n \geq 1$ in the above. But, because of reflection symmetry and reality of the Hamiltonian \mathcal{H} , we have

$$X_{-n}(t) = X_n(t) = X_n(-t)^*, \tag{64}$$

where the asterisk denotes complex conjugation.

The results in (63) are sufficient by themselves for $B = J$. But, also for $B \neq J$ we know the initial conditions now to high precision from earlier work and the newly derived asymptotic expansions for large n in Sect. 2. From these results we can find as many time-derivatives at $t = 0$ as we want using the differential equations (37) and (38). We can then calculate $X_n(\delta t)$ and $\dot{X}_n(\delta t)$ and their duals if $B \neq J$, to high precision using Taylor expansion to five orders for sufficiently small δt , or even more orders. Repeating this process N times we can calculate $X_n(N\delta t)$ and $\dot{X}_n(N\delta t)$ (and their duals) using initial conditions in the ‘‘past light-cone.’’ This is a discrete version of the method of characteristics. This is appropriate, since (37) and (38) are nonlinear differential-difference equations of hyperbolic type. In the continuum limit and for $B = J$, $\log X_n(t)$ satisfies the two-dimensional Klein–Gordon equation [13]. In the more general continuum scaling limit $B \rightarrow J$ we obtain the hyperbolic sine-Gordon equation [13, 25].

4.1 The Case of Critical Transverse Field $B = J$

When $B = J$, we must solve (38) with initial conditions (63). It is easily checked that $X_n(t)$ only depends on n and $\bar{t} \equiv Jt$. Equivalently we can choose the unit of time such that $J = 1$ and $\bar{t} \equiv t$, which we shall do in this subsection.

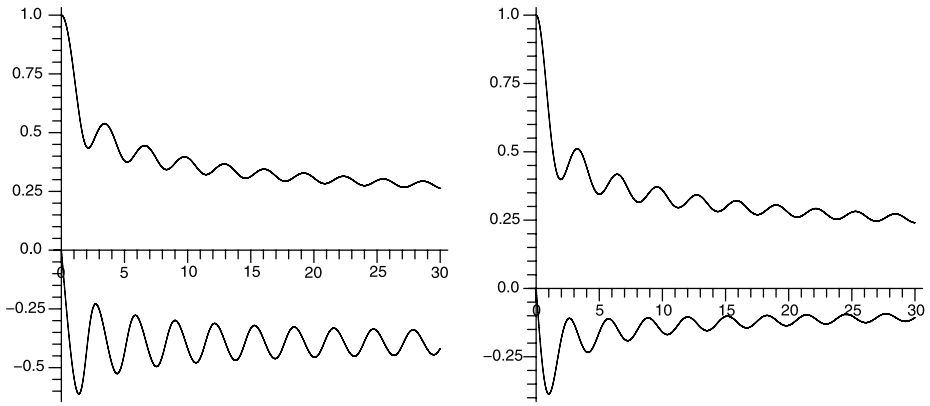


Fig. 1 $X_0(t)$ for $B = J = 1$. On the *left* are plotted $|X_0(t)|$ and $\arg X_0(t)$, while on the *right* $\Re X_0(t)$, and $\Im X_0(t)$ are shown

It is also convenient to work with the new variables

$$\xi_n(t) \equiv \log X_n(t), \quad \dot{\xi}_n(t) \equiv \frac{d\xi_n(t)}{dt} = \frac{\dot{X}_n(t)}{X_n(t)}, \tag{65}$$

$$\eta_n(t) \equiv \frac{X_{n+1}X_{n-1}}{X_n^2} = \exp(\xi_{n+1} + \xi_{n-1} - 2\xi_n). \tag{66}$$

Knowing these one can get the higher time-derivatives from (38), i.e.,

$$\ddot{\xi}_n(t) = \eta_n - 1, \tag{67}$$

$$\ddot{\xi}_n(t) = \eta_n(\dot{\xi}_{n+1} + \dot{\xi}_{n-1} - 2\dot{\xi}_n), \tag{68}$$

$$\ddot{\xi}_n(t) = \eta_n(\ddot{\xi}_{n+1} + \ddot{\xi}_{n-1} - 2\ddot{\xi}_n + (\dot{\xi}_{n+1} + \dot{\xi}_{n-1} - 2\dot{\xi}_n)^2), \tag{69}$$

$$\begin{aligned} \ddot{\xi}_n(t) = \eta_n & \left(\ddot{\xi}_{n+1} + \ddot{\xi}_{n-1} - 2\ddot{\xi}_n \right. \\ & + 3(\ddot{\xi}_{n+1} + \ddot{\xi}_{n-1} - 2\ddot{\xi}_n)(\dot{\xi}_{n+1} + \dot{\xi}_{n-1} - 2\dot{\xi}_n) \\ & \left. + (\dot{\xi}_{n+1} + \dot{\xi}_{n-1} - 2\dot{\xi}_n)^3 \right), \tag{70} \end{aligned}$$

and we can then calculate $\xi_n(t + \delta t)$ and $\dot{\xi}_n(t + \delta t)$ in Taylor series in δt to fifth, respectively fourth order.

We have implemented this on a Macintosh computer using GNU Fortran with $\delta t = 10^{-4}$. Starting at $t = 0$ we did $N = 300,000$ time steps, going up to $t = 30$. Therefore, we calculated $X_n(t)$ in the triangle in the n - t plane with the three corners (n, t) given by $(\pm 6 \times 10^5, 0)$ and $(0, 30)$. Some of the results are given in Fig. 1. Comparing the Fortran output for $X_0(t)$ with the asymptotic expansion (59), we concluded that we have already about 12 place agreement for $t > 20$, even though the Fortran program only worked with 15 place floating point complex numbers. This is as good as we could have hoped for and it demonstrates the stability of the procedure. It is a strong indication that all the Fortran output within the ‘‘characteristic triangle’’ is equally accurate.

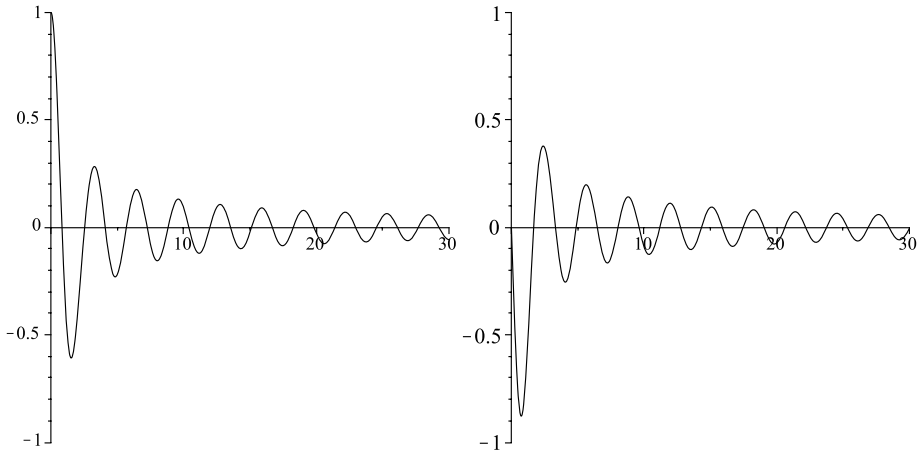


Fig. 2 $Y_0(t)$ is plotted for $B = J = 1$. On the left $\Re Y_0(t)$ is shown and on the right $\Im X_0(t)$

Next, we can use the general identities of Lajzerowicz and Pfeuty [68],

$$C_n(t) \equiv \langle \sigma_j^x(t) \sigma_{j+n}^y \rangle = -\langle \sigma_j^y(t) \sigma_{j+n}^x \rangle = \frac{1}{B} \frac{dX_n(t)}{dt}, \tag{71}$$

$$Y_n(t) \equiv \langle \sigma_j^y(t) \sigma_{j+n}^y \rangle = -\frac{1}{B^2} \frac{d^2 X_n(t)}{dt^2}, \tag{72}$$

which are valid for all J and B and also for finite temperatures. These quantities are also calculated in the Fortran program with $B = J = 1$ as

$$C_n(t) = \dot{\xi}_n(t) e^{\xi_n(t)}, \quad Y_n(t) = -\left(\ddot{\xi}_n(t) + \dot{\xi}_n(t)^2 \right) e^{\xi_n(t)}. \tag{73}$$

Their asymptotic expansions for $n = 0$ follow immediately from (59) and agree with the Fortran data with a similar accuracy as the $X_0(t)$ did. We have plotted $Y_0(t)$ in Fig. 2.

Finally, we remark that we could have made the plots using only the Painlevé V equation of [26–29]. We would, however, not have been able to calculate the about 10^{11} other very accurate data points within the “characteristic triangle” that way.

4.2 The Case of Noncritical Transverse Field $B \neq J$

In this subsection we assume that $J < B$ in our transverse-field Ising chain, so that $B^* < J^*$ for the dual Ising chain with $B^* \equiv J$ and $J^* \equiv B$. We choose time units such that $B = J^* = 1$,¹¹ whence $J = B^* = k$ and (37) is to be replaced by

$$\begin{aligned} X_n(t) \ddot{X}_n(t) - \dot{X}_n(t)^2 &= X_{n-1}^*(t) X_{n+1}^*(t) - X_n^*(t)^2, \\ X_n^*(t) \ddot{X}_n^*(t) - \dot{X}_n^*(t)^2 &= k^2 \left(X_{n-1}(t) X_{n+1}(t) - X_n(t)^2 \right). \end{aligned} \tag{74}$$

¹¹Equivalently, we may replace t by $\bar{t} \equiv Bt = J^*t$, $k = J/B$ in the following formulae.

At the critical point $T = 0, B = J$, we found it better to work with the logarithm $\xi_n(t)$ of the pair correlation function $X_n(t)$. Once we leave the critical point it becomes even essential to do so because of the exponential decay of the connected pair correlation with separation n , as we need initial conditions for very large n . Therefore, we introduce $\xi_n(t)$ and $\xi_n^*(t)$ via

$$X_n(t) = (1 - k^2)^{1/4} e^{\xi_n(t)}, \tag{75}$$

$$X_n^*(t) = (1 - k^2)^{1/4} (1 + e^{\xi_n^*(t)}), \tag{76}$$

Substituting these into (74) these equations become

$$\ddot{\xi}_n(t) = e^{\xi_{n-1}^* - 2\xi_n} + e^{\xi_{n+1}^* - 2\xi_n} - 2e^{\xi_n^* - 2\xi_n} + e^{\xi_{n-1}^* + \xi_{n+1}^* - 2\xi_n} - e^{2\xi_n^* - 2\xi_n}, \tag{77}$$

$$\ddot{\xi}_n^*(t) = \frac{k^2(e^{\xi_{n-1} + \xi_{n+1} - \xi_n^*} - e^{2\xi_n - \xi_n^*}) - \dot{\xi}_n^{*2}}{1 + e^{\xi_n^*}}. \tag{78}$$

These expressions were written such that none of the real parts of their exponents goes rapidly to $+\infty$.

The initial conditions for $\xi_n(t)$ and $\xi_n^*(t)$ are determined by (29), (30), (60), (61), and (62). One immediately sees that $\xi_n(0)$ and $\xi_n^*(0)$ roughly grow linearly with $|n|$, which is much better for the numerics than the exponential behavior of $X_n(0)$ and $X_n^*(0)$. Setting, as explained above, $B = 1$ in (60) for $X_n(0)$ and $B^* = k$ for $X_n^*(0)$, we find

$$\dot{\xi}_n(0) = -\frac{2iE(k)}{\pi} \delta_{n0}, \tag{79}$$

$$\dot{\xi}_n^*(0) = -\frac{2i(E(k) - (1 - k^2)K(k))}{\pi(1 - (1 - k^2)^{1/4})} \delta_{n0}. \tag{80}$$

The higher derivatives at $t = 0$ can then be evaluated using the pair of coupled differential equations (77) and (78).

We can now adopt the same strategy as for the critical-field case $B = J$. Given $\xi_n(t)$ and $\xi_n^*(t)$ and their first five time-derivatives at a given t , we can calculate $\xi_n(t + \delta t)$ and $\xi_n^*(t + \delta t)$ in a Taylor series through fifth order in δt and $\dot{\xi}_n(t + \delta t)$ and $\dot{\xi}_n^*(t + \delta t)$ in a Taylor series through fourth order. We have implemented this in a Maple program with $\delta t = 10^{-3}$ going up to $t = 8.77$.

Our results for $X_0(t)$ at $k = 0.7$ are represented in Fig. 3, using both our Maple numerical integration results for $t \leq 8.77$ and our asymptotic expansions (51) for $t > 5.77$. There is clearly excellent agreement between the numerical integration and the asymptotic expansion.¹²

We have also plotted $X_0(t)$ and $Y_0(t)$ for $t \leq 50$ using the numerical results for $t \leq 8.77$ and the asymptotic expansions for $t \geq 8.77$. We have again used (72) to calculate $Y_0(t)$. Note that the lowest frequency, clearly seen in Fig. 4 for $X_0(t)$ is hardly visible after two time derivatives in Fig. 5. The ratio of the two leading frequencies is $3/17$, as follows from (51), leading to a suppression factor of $9/289$.

¹²We note that there is no agreement with the corresponding lowest-order asymptotic results in [11] because of several misprints there.

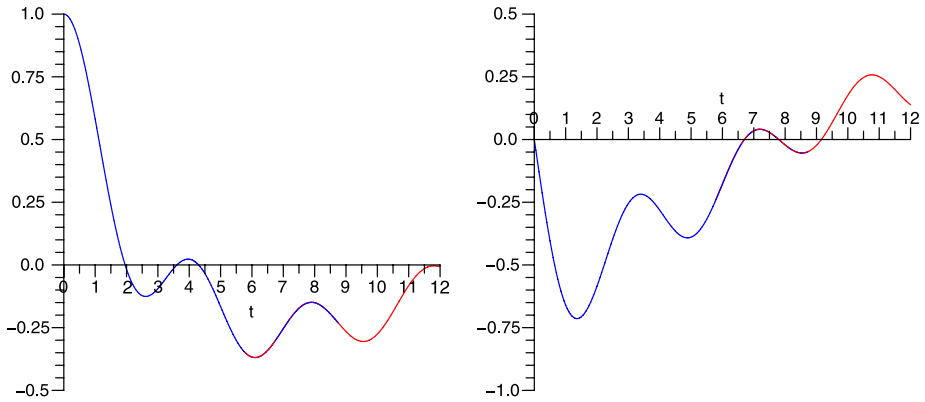


Fig. 3 (Color online) $X_0(t)$ for $B = 0.7, J = 1$: $\Re X_0(t)$ and $\Im X_0(t)$

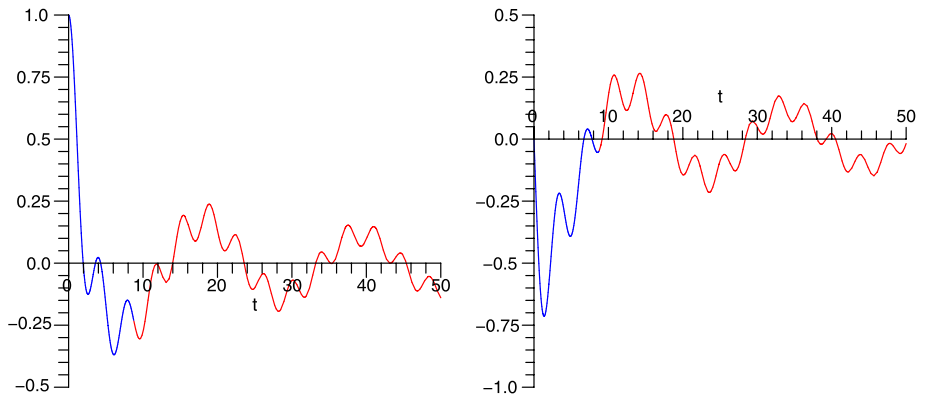


Fig. 4 (Color online) $X_0(t)$ for $B = 0.7, J = 1$: $\Re X_0(t)$ and $\Im X_0(t)$

Because of the very nature of the differential equations of [12] used, we also got the results for the dual model from the same Maple run. They correspond to $X_n(t)$ at $B/J = 1/k$, or $J/B = 0/7$. We have plotted $X_0(t)$ for $t \leq 12$ in Fig. 6. Again it is seen that the numerical results approach the asymptotic behavior well. Note that the vertical scales are smaller than in Fig. 3, making the small differences more pronounced. We have also plotted $X_n(t)$ and $Y_n(t)$ for $t \leq 50$ for the same values of the parameters (Figs. 7 and 8). Unlike Fig. 4, there is now no pronounced two-frequency behavior, as other frequencies are down two orders in t^{-1} in (50). In addition, it is clearly seen that $X_0(t)$ decays to the square of the order parameter $(1 - k^2)^{1/4} \approx 0.845$, as it should [67].

5 Final Remarks

The original motivation for this work came from two sides. First, we were told by some colleagues that solving the equations of [12] numerically leads to useless results inconsistent with earlier results, especially [11]. We believe that this point of view has been put to rest by

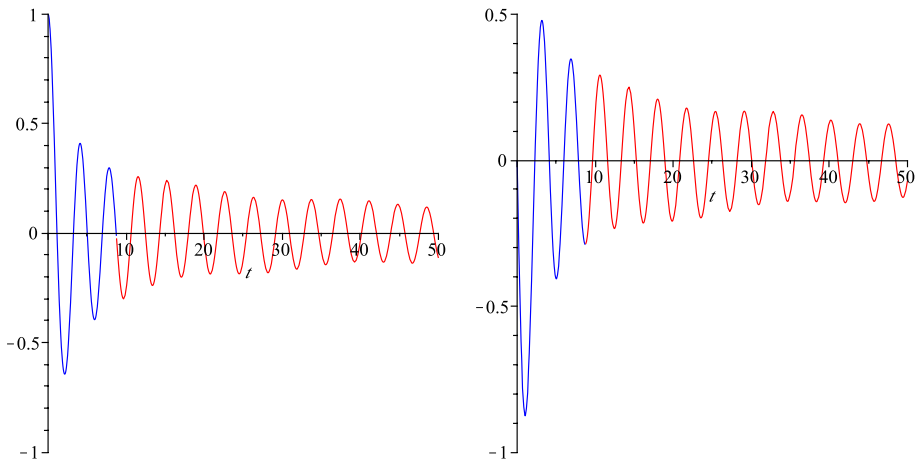


Fig. 5 (Color online) $Y_0(t)$ for $B = 0.7, J = 1$: $\Re Y_0(t)$ and $\Im Y_0(t)$

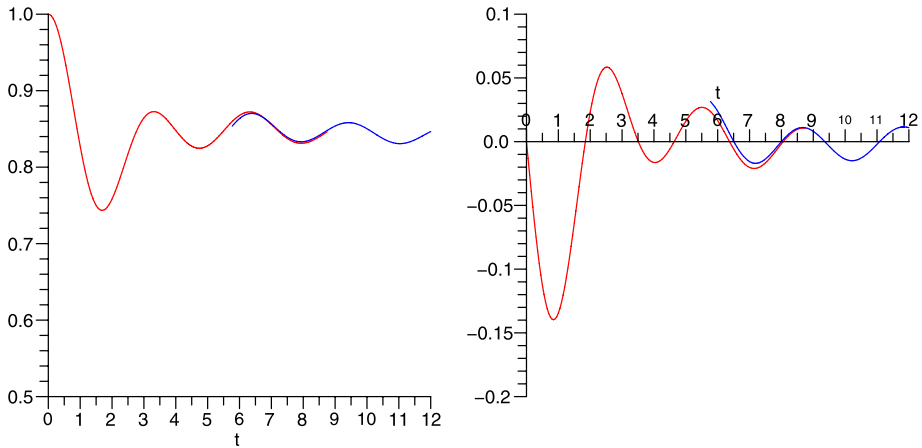


Fig. 6 (Color online) $X_0(t)$ for $J = 0.7, B = 1$: $\Re X_0(t)$ and $\Im X_0(t)$

the results in this paper. Secondly, during a brief visit in 2005 at the University of New South Wales one of us got to discuss an early version of [69, 70] with his colleagues. It led us to calculate the small and large- B expansions for the structure function $\langle \sum_j \sigma_0^x \sigma_j^x \rangle$ of the Ising chain in a transverse field at zero temperature. This turned out to be nothing but the high- and low-temperature series of the “diagonal susceptibility” of the two-dimensional Ising model $\beta \sum_n \langle \sigma_{0,0} \sigma_{n,n} \rangle$, the calculation of which is a small part of the calculation of [50, 51]. A different approach to the diagonal susceptibility in terms of form factors and n -particle contributions has been given in [71, 72] revealing a lot of its mathematical structure.

With the results of this paper it is now also very easy to calculate extremely accurate values of the q -dependence of the structure function [69, 70] and of the second moment correlation length (as used e.g. in [73]).

Furthermore, it is well-known that the correlation functions of the zero-field XY-model factor into two correlation functions of the Ising model in transverse field [20, 27], just like

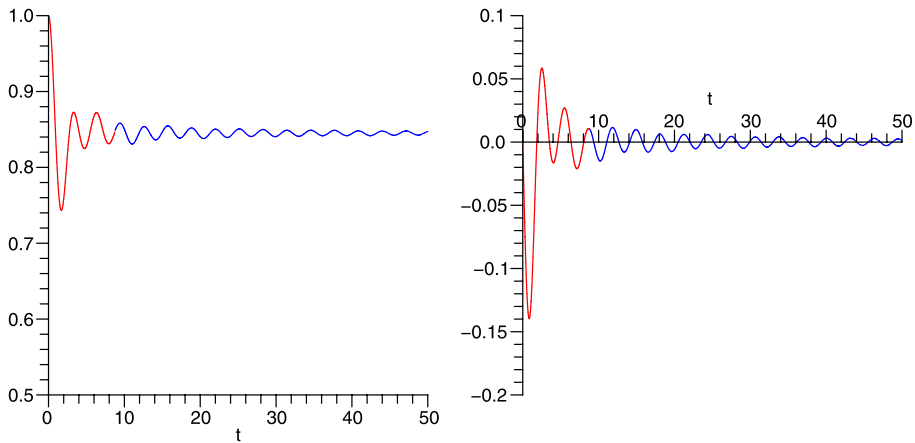


Fig. 7 (Color online) $X_0(t)$ for $J = 0.7$, $B = 1$: $\Re X_0(t)$ and $\Im X_0(t)$

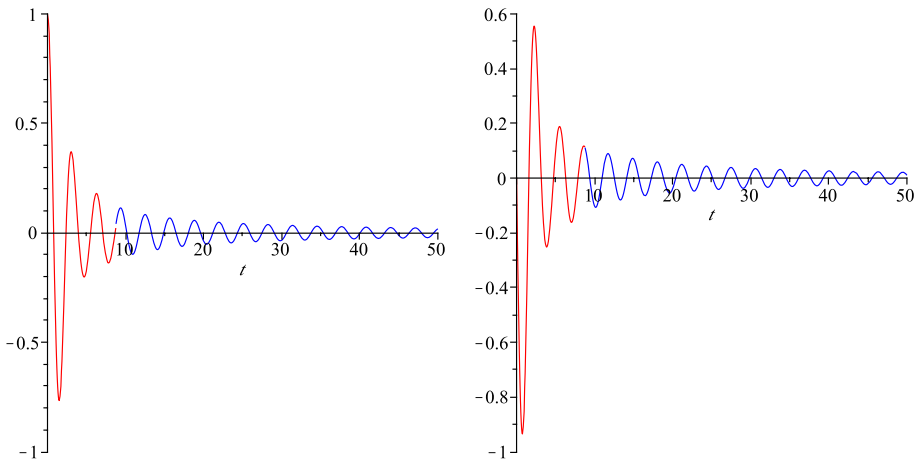


Fig. 8 (Color online) $Y_0(t)$ for $J = 0.7$, $B = 1$: $\Re Y_0(t)$ and $\Im Y_0(t)$

correlations in the two-dimensional square-lattice dimer and fully-frustrated Ising models factor [74, 75]. Therefore, our results directly apply also to those cases.

Finally, as more and more results on time-correlations in the more general XXZ model have become available [76–83], one may express the hope that some new powerful identities will be discovered also for this case that will facilitate numerical calculations dramatically.

Acknowledgements First of all, we must thank the people at the Mathematics and Theoretical Physics Departments at Australian National University, where this work was finished, for their hospitality and support. We are also grateful to Dr. C.J. Hamer, Dr. J. Oitmaa and Dr. W. Zheng of the University of New South Wales, where the original concept of this paper was birthed during a brief visit in July, 2005. Part of this work has been performed on a new computer provided by Dean Dr. P.M.A. Sherwood of Arts and Sciences at Oklahoma State University.

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